Lagrangians and Fields

Independent Lagrangians

$$\mathcal{L} = \mathcal{L}_x(x, \dot{x}) + \mathcal{L}_y(y, \dot{y}), \qquad (5.1)$$

$$\mathcal{L} = \frac{1}{2}(\dot{x})^2 + \frac{1}{2}(\dot{y})^2 - V_x(x) - V_y(y),$$

Dependent Lagrangian

$$\mathcal{L} = \frac{1}{2}(\dot{x})^2 + \frac{1}{2}(\dot{y})^2 - V_x(x) - V_y(y) + xy.$$

Particle's Lagrangian

$$\mathcal{L}_{particle} = -m\sqrt{1-\dot{x}^2}.$$

Particle + Field Lagrangian

$$\int \mathcal{L}_{particle} dt = \int -[mc^2 + g\phi(t,x)]\sqrt{1-\frac{\dot{x}^2}{c^2}}dt.$$

$$\int \mathcal{L}_{particle} dt = \int -[mc^2 + g\phi(t,x)]\left(1-\frac{\dot{x}^2}{2c^2}\right)dt. \qquad (5.2$$

Use approximation $(1-\epsilon)^{1/2} \approx 1 - \frac{\epsilon}{2}$

$$Action_{particle} = \int -[m + g\phi(t,x)]\left(1-\frac{\dot{x}^2}{2}\right)dt. \qquad (5.5$$

$$\mathcal{L}_{particle} = \frac{m\dot{x}^2}{2} - g\phi(t,x). \qquad (5.3)$$

Let $c \to \infty$, then matches $T - V$ pattern

g indicates the strength of the coupling (i.e. gravity constant or electric charge)

What we're studying is a combined system that consists of a) a field and b) a particle moving through the field.

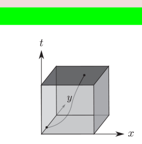

Figure 5.1: Particle moving through a region of spacetime filled with "red mush"—the scalar field $\varphi(t, x)$.

Contravariant & Co Vectors

Lorentz Contra & Co Vectors

$$(A')^t = \frac{A^t - vA^x}{\sqrt{1-v^2}}$$

$$(A')^x = \frac{A^x - vA^t}{\sqrt{1-v^2}}.$$

$$(A')_t = \frac{A_t + vA_x}{\sqrt{1-v^2}}.$$

$$(A')_x = \frac{A_x + vA_t}{\sqrt{1-v^2}}.$$

Contravariant

$$d\phi(x) = \frac{\partial\phi(x)}{\partial X^\mu}dX^\mu. \qquad (5.24)$$

$d\phi(x)$ is scalar making $\frac{\partial\phi(x)}{\partial X^\mu}$ a 4-vector

$$\frac{\partial\phi(x)}{\partial X^\mu} = \left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right).$$

$$\partial_\mu\phi = \frac{\partial\phi(x)}{\partial X^\mu}.$$

$$\partial_\mu\phi \Longleftrightarrow \left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$\partial^\mu\phi \Longleftrightarrow \left(-\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right).$$

$$\partial^\mu\phi\partial_\mu\phi = -\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2.$$

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi.$$

Equations of Motion

Euler-Lagrange

$$\sum_\mu \frac{\partial}{\partial X^\mu}\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial X^\mu}\right)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0. \qquad (5.15)$$

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2} = -g\rho(x). \qquad (5.16)$$

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2} = -g\delta^3(x). \qquad (5.17)$$

$$\nabla^2\phi = g\delta^3(x). \qquad (5.18)$$

Static solution (i.e. $\frac{\partial^2\phi}{\partial t^2} = 0$)

$$-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = g\delta(x - a(t)).$$

Add back in time dependence and allow ϕ to move as a function of time

Particle Affects Field

$$Action_{field} = \int \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 dx. \qquad (5.4)$$

$$Action_{particle} = \int -[m + g\phi(t,x)]\left(1-\frac{\dot{x}^2}{2}\right)dt. \qquad (5.5$$

$$Action_{interaction} = \int -g\phi(t,0)dt. \qquad (5.6)$$

Set $x = 0$ and $\dot{x} = 0$

$$Action_{interaction} = \int -g\phi(t,x)\delta^3(x)d^4x. \qquad (5.12$$

Use source function $\rho(x) = \delta^3(x)$

$$Action_{total} = \int \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - g\delta^3(x)\phi(t,x)\right]d^4x$$

$$\mathcal{L}_{total} = \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - g\rho(x)\phi(t,x)\right]. \qquad (5.14$$

Waves

Waves

$$\mathcal{L} = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}\right)^2 - \left(\frac{\partial\phi}{\partial y}\right)^2 - \left(\frac{\partial\phi}{\partial z}\right)^2 - \mu^2\phi^2\right]. \qquad (5.26$$

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2} + \mu^2\phi = 0. \qquad (5.27)$$

$$\phi = e^{-i\omega t}e^{i(k_x x + k_y y + k_z z)}, \qquad (5.28)$$

$$e^{-i\omega t}e^{i(k_x x + k_y y + k_z z)} = e^{i(k_\mu X^\mu)}. \qquad (5.29)$$

$$\omega = \pm\sqrt{{k_x}^2 + {k_y}^2 + {k_z}^2 + \mu^2}. \qquad (5.30)$$

$$\phi(x) = e^{i(kx - \omega t)}$$

$$e^{i(kx-\omega t)} = \cos(kx - \omega t) + i\sin(kx - \omega t), \qquad (5.25$$

$$\phi = e^{-i\omega t}e^{i(k_x x + k_y y + k_z z)}, \qquad (5.28)$$

4 Vectors

4-Vector

$$X^\mu = (t, x, y, z),$$

Metric

Calling it a 4-vector is a statement about the way it behaves under Lorentz transformation; any complex of four quantities that transforms the same way as t and x is a 4-vector

A statement that all four components of a 4-vector are zero is an invariant statement

If a scalar is equal to zero in one frame of reference, it's zero in every frame. Indeed, this is the definition of a scalar; it's a thing that's the same in every frame.

Metric

$$-(A^t)^2 + (A^x)^2 + (A^y)^2 + (A^z)^2$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$\sum_\nu \eta_{\mu\nu}A^\nu = \eta_{\mu 0}A^0 + \eta_{\mu 1}A^1 + \eta_{\mu 2}A^2 + \eta_{\mu 3}A^3.$$

$$A_\mu = \sum_\nu \eta_{\mu\nu}A^\nu.$$

$$A_\nu = \begin{pmatrix} -A^t \\ A^x \\ A^y \\ A^z \end{pmatrix}.$$

$$A_\mu = \eta_{\mu\nu}A^\nu \qquad (5.20)$$

$$A_\mu = \eta_{\mu\nu}A^\nu \qquad (5.20)$$

$$A^\mu A_\mu = A^\mu\eta_{\mu\nu}A^\nu. \qquad (5.21)$$

$$A^\mu A_\mu \qquad (5.19)$$

Ex. 5.1

$$A^\nu A_\nu = \sum_i A^i A_i = \sum_i A_i A^i = A_\nu A^\nu$$

Ex. 5.2:

$$A_\mu = \nu_{\mu\nu}A^\nu$$

$$\Longrightarrow A^\nu = \nu_{\mu\nu}\nu_{\mu\nu}A^\nu$$

$$(A+B)^\mu(A+B)_\mu - (A-B)^\mu(A-B)_\mu = 4\left[A^\mu B_\mu\right].$$

You can think of AμBμ as the Lorentz or Minkowski version of the dot product.